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Gravity as a bend of 4D elastic plate

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Abstract

Gravity is represented as a linear theory of strong plate bend at the variational functionals level. Some estimates of elastic constant of space-time are made. Physical (elastic) sense of field lagrangians and Einstein Equations is discussed.

The question about physical nature of space, time, gravity and Einstein Equations (EE) has been discussing from the earliest era of special (SR) and general (GR) relativity up to a present time. Today we have powerful mathematical means of formulation and investigation of space-time physics, while physical foundations of the theories and their relations both to observable world and to other physical topics often remain beyond the scope of attention.

In my report I am going to demonstrate physical relevance of linear elasticity theory (Hooks law) language for formulation and clarifying standard GR. Note, that difficulties of most of the attempts, that have been made in this direction, were caused by restriction to (at best) 4D elasticity([1]-[3]) — we'll see that it totally excludes the possibility of "embedding" gravity into elasticity.

The starting points of our consideration are two quite obvious observations:

1) We locally live in *4D Minkowski world* $\mathbb{M}_4$;

2) In every 3D simultaneity section, which is locally isometric to 3D euclidian space $\mathbb{E}_3$, we deal with (approximate) *Hooks law,* when investigate contact interactions properties of real mechanical bodies:

$$\sigma = 2\mu\mathcal{D} + \lambda \mathrm{Tr}[\mathcal{D}]\eta, \tag{1}$$

where σ and $\mathcal{D}$ — 3D *stress* and *strain tensors,* defined by expressions[1]:

$$\mathcal{D} \equiv \frac{1}{2}\left[(\overrightarrow{\partial} \otimes \xi + \xi \otimes \overleftarrow{\partial}) + (\partial \otimes \partial)\eta(\xi, \xi)\right]; \quad \sigma = \frac{\partial \mathfrak{F}}{\partial \mathcal{D}}, \tag{2}$$

$\xi = \xi(x)$ — *displacements vector field,* $\mathfrak{F}$ — *elastic energy density*:

$$\mathfrak{F} = \mu\mathcal{D}^2 + \frac{\lambda}{2}\mathrm{Tr}^2[\mathcal{D}], \tag{3}$$

μ, λ — pair of independent elastic constants — *Lame coefficients,* η — euclidian metric in $\mathbb{E}_3$ [8, §4]. There are many different generalizations of these two points that can be assumed to reveal physical properties of space, time and gravity. Now we accept the most simple ones ([4]-[5]):

1) We live on *pseudoeuclidian 4D plate,* i.e. such multidimensional body $\mathcal{P}$, whose sizes along some $\mathbb{M}_4$ are much larger, than along orthogonal to $\mathbb{M}_4$ dimensions:

$$\mathcal{P} \simeq \mathbb{M}_4 \times \overline{\mathbb{M}}_N \simeq \mathbb{M}_4 \mathop{\times}_{i=1}^{N} \left[-\frac{h_i}{2}; \frac{h_i}{2}\right] \subset \mathbb{M}_{N+4},$$

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[1] I'll use indexless representation. Particularly, ∂ and ∂^2 will denote partial derivative operators of first and second orders respectively.

where "$\simeq$" denotes homeomorphism relation, $\{h_i\}$ $(i = \overline{1, N})$ — the set of small thicknesses of the plate in extradimensions, $\mathbb{M}_{N+4}$ — embedding space-time.

2) This 4D plate possesses isotropic elastic properties, which we'll describe by the same denoted pair of Lame coefficients λ and μ in $N + 4$-dimensional Hooks law:

$$\sigma = 2\mu\mathcal{D} + \lambda \mathrm{Tr}[\mathcal{D}]\Theta, \tag{4}$$

that generalizes (1) ($\Theta = \theta + \bar{\theta}$ in (4) — metrics on $\mathbb{M}_{N+4}$, $\mathbb{M}_4$ and $\overline{\mathbb{M}}_N$ correspondingly and all other notations are related to the multidimensional plate). The (4) determines multidimensional elastic energy density, having the same form as in (3), which we intend to calculate.

Imagine that the plate in unstrained state is placed in $\mathbb{M}_{N+4}$, occupying a subspace defined by the equation: $\bar{x} = 0$, where $\{\bar{x}\}$ together with $\{x\}$ form $(N + 4)$-dimensional Cartesian coordinate system: $\{X\} = \{x\} \times \{\bar{x}\}$, adjusted to the plate surface $\mathbb{M}_4$ (so $\partial_x \in T\mathbb{M}_4$, $\partial_{\bar{x}} \in T\mathbb{M}_N$). Then its mechanical straining can be described by a smooth displacement vector field Ξ: $\mathcal{P} \xrightarrow{\Xi} \mathcal{P}'$, that takes in our coordinate frame the form $X \to X' = X + \Xi$ or in projections $x' = x + \xi$; $\bar{x}' = \bar{x} + \bar{\xi}$, where $P(X) \in \mathcal{P} \to P(X') \in \mathcal{P}'$ — some arbitrary matter point inside the body, $\Xi = \xi + \bar{\xi}$, $\xi \in T\mathbb{M}_4$, $\bar{\xi} \in T\mathbb{M}_N$.

It is well known, that within linear elasticity theory deformations normal and tangent to the plate are factorizable: their contributions into elastic energy density can be calculated separately and independently [8, §11,§13]. Let us start from the

1. Bend deformation. We have pure bend without stretching with $\xi = 0$, $\partial^2\bar{\xi} \neq 0$. Suppose, that middle (in thicknesses) plane $\mathbb{M}_4$ of the unstrained plate is transformed by the bending $\bar{\xi}$ into some Riemannian manifold $\mathbb{V}_4$. This new middle surface in case of a weak bend is usually called *neutral surface* because tangent stresses $\sigma|_{\mathbb{V}_4} = 0$. Since both thicknesses and bend are small we can omit any bending external force densities in comparison with internal stresses (similarly to ordinary 2-D plates in 3D euclidian space [8, §11]) and also omit variations of $\bar{\partial}$: $\bar{\partial}' \approx \bar{\partial}$, where $\bar{\partial} \equiv \partial_{\bar{x}}$. It implies $\sigma(\bar{\partial},\ ) \approx 0$ at every point of $\mathcal{P}$. Then from (4) we get the following algebraic equations for strain tensor (and differential equations for displacements field):

$$\mathcal{D}(\bar{\partial}_i, \bar{\partial}_j) = \mathcal{D}(\partial, \bar{\partial}) = 0; \quad 2\mu\mathcal{D}(\bar{\partial}_i, \bar{\partial}_i) + \epsilon_i \lambda \mathrm{Tr}[\mathcal{D}] = 0,$$

where $\epsilon_i = \pm 1$ if $\bar{\partial}_i$ is time-like or space-like respectively. Its solution is $\xi = -\bar{\theta}(\bar{x}, \partial\bar{\xi})$. Then for $\mathcal{D}$ we obtain $(N\lambda + 2\mu \neq 0)$:

$$\mathcal{D}(\partial, \partial) = -\bar{\theta}(\bar{x}, \partial^2\bar{\xi}); \quad \mathcal{D}(\bar{\partial}_i, \bar{\partial}_i) = -\epsilon_i \frac{\lambda S}{N\lambda + 2\mu}; \quad S \equiv \mathrm{div}\,\xi \equiv \theta(\partial, \xi) = -\bar{\theta}(\bar{x}, \Box\bar{\xi}). \tag{5}$$

Substituting (5) into (3) we get

$$\mathfrak{F}_{\mathrm{b}} = \mu \left\{ \bar{\theta}^2(\bar{x}, \partial^2\bar{\xi}) + f\, \bar{\theta}^2(\bar{x}, \Box\bar{\xi}) \right\}, \tag{6}$$

where factor $f = \lambda/(N\lambda + 2\mu)$.

To obtain expression for total bend energy of the plate one should integrate (6) over its $N + 4$-dimensional volume:

$$F_{\mathrm{b}} = \frac{\mu H_N}{12} \int\limits_{\mathbb{M}_4} \left\{ \bar{\theta}_{\mathrm{h}}(\partial^2\bar{\xi}, \partial^2\bar{\xi}) + f\, \bar{\theta}_{\mathrm{h}}(\Box\bar{\xi}, \Box\bar{\xi}) \right\} d\mathrm{m}[\mathbb{M}_4], \tag{7}$$

where measure $d\mathrm{m}[\mathbb{M}_4]$ on $\mathbb{M}_4$ is defined from the decomposition $d\mathrm{m}[\mathbb{M}_{N+4}] = d\mathrm{m}[\mathbb{M}_4] \cdot d\mathrm{m}[\mathbb{M}_N]$,

$$\frac{\bar{\delta}_{\mathrm{h}} H_N}{12} = \int\limits_{\mathbb{M}_N} (\bar{x} \otimes \bar{x})\, d\mathrm{m}[\mathbb{M}_N].$$

The above equation has been used with $H_N \equiv \prod\limits_{i=1}^{N} h_i$, $\bar{\delta}_h = \mathrm{diag}(h_1, \ldots, h_N)$, $\bar{\theta}_{\mathrm{h}} \equiv \bar{\delta}_{\mathrm{h}} \cdot \bar{\theta} = \mathrm{diag}(\epsilon_1 h_1, \ldots, \epsilon_N h_N)$. From the view point of GR, expression (7) should be relevant to gravity far from its source, where bending is weak. Near the sources we need to take into account tangent stretches and shears of the plate (i.e. energy-momentum tensors of the sources) even within the linear elasticity.

2. 4D stretches and shears. 4D elastic shears and stretches energy, that is generated by a strong bend, has the form:

$$F_{\rm s} = H_N \int\limits_{\mathbb{M}_4} \mathfrak{F}_{\rm s}\, d{\rm m}[\mathbb{M}_4] = \frac{H_N}{2} \int\limits_{\mathbb{M}_4} (\sigma \cdot \mathcal{D})\, d{\rm m}[\mathbb{M}_4], \tag{8}$$

where the standard representation of elastic energy density of tangent deformation in linear elasticity theory [8]: $\mathfrak{F}_{\rm s} = (\sigma \cdot \mathcal{D})/2 = \sigma_{\alpha\beta}\mathcal{D}^{\alpha\beta}/2\ (\alpha, \beta = \overline{0,3})$ has been used. So, full variational functional F, including potential energy of bending and stretching multidimensional forces external to the plate, takes the form

$$F = F_{\rm b} + F_{\rm s} + U. \tag{9}$$

3. Comparing with Einstein-Gilbert action of GR. Now we compare action (9) with full action $S = S_{\rm g} + S_{\rm m}$ of a system "gravitational field + matter source" in GR. To reveal the similarity of structures of $F_{\rm b}$ and $S_{\rm g}$ we need to reformulate the latter in terms of embedding theory in deformation representation, where any Riemannian manifolds $\mathbb{V}_4$ is interpreted as Minkowski plane $\mathbb{M}_4$, deformed by some normal to the plate vector field $\bar{\xi}(x)$. Induced Riemannian metric g will then be

$$g = \theta + 2\mathcal{D}, \tag{10}$$

where $\mathcal{D}$ is the second (nonlinear) term in (2) and GR action in this representation is:

$$S_{\rm g} = -\frac{c^3}{16\pi G} \int\limits_{\mathbb{V}_4} {}^{(4)}\mathcal{R}\, d{\rm m}[\mathbb{V}_4] = -\frac{c^3}{16\pi G} \int\limits_{\mathbb{V}_4} \left\{ \mathcal{H}(\Box\bar{\xi}, \Box\bar{\xi}) - \mathcal{H}(\partial^2\bar{\xi}, \partial^2\bar{\xi}) \right\}\, d{\rm m}[\mathbb{V}_4], \tag{11}$$

where ${}^{(4)}\mathcal{R}$ — scalar curvature of $\mathbb{V}_4$, $\mathcal{H} \stackrel{\rm def}{\equiv} \sum_{m=1}^{N} \epsilon_m \cdot n_{(m)} \otimes n_{(m)}$ — projector on subspace of $T\mathbb{M}_{N+4}$ orthogonal to $\mathbb{V}_4$, $\{n_{(m)}\}$ — basis vector fields of the subspace, normalized by conditions: $\Theta(n_{(m)}, n_{(l)}) = (\bar{\theta}_{\rm h})_{ml}$. Note, that under the bend in linear approximation (when Hooks law is valid), we have $\mathcal{H} \approx \bar{\theta}_h$, and $d{\rm m}[\mathbb{V}_4] \approx d{\rm m}[\mathbb{M}_4]$ Then under $f = -1$ expressions (7) for $F_{\rm b}$ and (11) for $S_{\rm g}$ become identical up to a dimensional constant. The remaining parts — $F_{\rm s}$ and $S_{\rm m}$ should be also identified, as those involving tangent to $\mathbb{V}_4$ stresses (energy-momentum tensors). There are no analogies of external energy U in standard physics, since it concerns noncausal (from the viewpoint of $\mathbb{V}_4$) interaction of the plate with its multidimensional environment.

4. Discussion.

Let us briefly discuss some general consequences of the approach.

Multidimensional elastic constants. It is easily to show that value of elastic parameter $f = -1$, reproducing integrand of (11) in (7), corresponds to the Poisson coefficient of the plate medium $\sigma = 1/2$ ([4]). Second independent elastic constant — Young modulus E — can be evaluated by some dimensional manipulations. The result is

$$Eh^{N+3} \sim \frac{c^4}{G} \sim \frac{1}{æ},$$

that supports old Sacharov's considerations on possible relation between Einstein constant and elastic properties of space-time [2].

Lagrange formalism as 4-D elasticity theory. Let us draw attention to analogy between these two expressions:

$$\delta F_{\rm s} = \int \sigma \cdot \delta\mathcal{D}\, d{\rm m}; \quad \delta S_{\rm m} = \frac{1}{2c} \int \mathcal{T} \cdot \delta g\, d{\rm m},$$

where dm denotes suitable form of volume of the base manifold. The first is general thermodynamic relation, connecting stress tensor with infinitisemal variation of an elastic free energy [8, §3]. The second — is well known rule for calculation of symmetric energy-momentum tensor $\mathcal{T}$ of fields or matter from its lagrangian density $\mathcal{L}$, $S_{\rm m}[q] = \int \mathcal{L}(q, \partial q)\, d{\rm m}$, where $\{q\}$ and $\{\partial q\}$ are sets of field variables and their derivatives respectively [9]. Our approach shows, that these expressions have deep interrelations, under assumption of the relation $F_{\rm s} = cS_{\rm m}$. Really, curvelinear metric g can be understood as result of tangent to $\mathbb{M}_4$ diffeomorphism $x \xrightarrow{\xi} x' = x'(x)$, which has not passive (as in GR), but active sense. Then in accordance with (10) metric on $\mathbb{M}'_4$ will be $g = \theta + 2\mathcal{D}$, where θ — metric on original $\mathbb{M}_4$. Both $\mathfrak{F}_{\rm s}$ and $\mathcal{L}$ implicitly contain metric g (to get scalar expressions from $\mathcal{D}$ or q, and ∂q). From the kind of g we get $\delta g = 2\delta\mathcal{D}$, and so

$$\sigma = \frac{\delta F_{\rm s}}{\delta \mathcal{D}} = 2\frac{\delta F_{\rm s}}{\delta g} = \frac{\delta S_{\rm m}}{\delta g} = \mathcal{T}. \tag{12}$$

If this analogy is not occasional, then *any classical field lagrangian can be interpreted as elastic energy density of some strain of* $\mathbb{M}_4$ *and specific choice of field variables is determined by kind of the straining:* $\mathcal{D} = \mathcal{D}(q)$ [5].

Physical essense of Einstein Equations. Extremality of $F[\Xi]$ and $S[q]$ leads to Euler-Lagrange equations, which in turn, provide validity of equilibrium equations in the first case and conservation laws in the second:

$$\delta F = 0 \longrightarrow \operatorname{div}\sigma = 0; \qquad \delta S = 0 \longrightarrow \operatorname{div} T = 0. \tag{13}$$

Within the present approach it would be natural to use unified language and regard conservation laws as equilibrium equation of some elastic body. In view of presented results, we conclude, that this body is *space-time itself.*

Assume, that space-time plate is characterized by "phenomenological" multidimensional elastic constant E and σ: $S_{\rm g} = S_{\rm g}(E, \sigma)$, where $S_{\rm g}$ — is generalized action for gravitational field, or free elastic energy of bending (7). Then

$$\delta_g S = \int (\mathcal{T}^{(n)}(E, \sigma) + \mathcal{T}^{(t)}) \cdot \delta g \, d{\rm m}. \tag{14}$$

where $\mathcal{T}^{(n)}$ is generated by normal straining of the plate, the $\mathcal{T}^{(t)}$ — by tangent ones. Vanishing of (14) gives

$$\mathcal{T}^{(n)}(E, \sigma) + \mathcal{T}^{(t)} = 0 \tag{15}$$

— "generalized" Einstein equations. One can conclude, that Einstein theory, even in this "generalized" variant, operates with *nonstressed* state of space-time. In other words, *physical meaning of standard Einstein equations is expressed in intercompensation of tangent stresses, generated by normal and tangent plate strains.* From the view point of elasticity true dynamical variables are components of strain vector Ξ and true equilibrium equations of the plate are:

$$\operatorname{div}(\mathcal{T}^{(n)}(E, \sigma) + \mathcal{T}^{(t)}) = 0. \tag{16}$$

The (16) are consequence of *the strong bend* equations:

$$\Box^2 \bar{\xi}_D - H_N \theta(\partial, \sigma(\ , \partial\bar{\xi})) = \bar{P}; \quad \text{and} \quad H_N \operatorname{div}\sigma = -P,$$

which are deduced by variational procudure over Ξ and so are true (necessary) equilibrium equations. Here $\{D\}$ — the set of *cylindrical stiffness* factors of the plate in extradimensions, $\Pi = P + \bar{P}$ — total, tangent, and normal to the plate surface multidimensional force densities respectively [5]. As it has been mentioned earlier, one can go to Einstein GR by setting $\sigma = 1/2$. Under this condition $\mathcal{T}^{(n)}(E, 1/2)$ transforms to

$$-G/æ = -({}^{(4)}\mathcal{R}{\rm ic} - (1/2) g\, {}^{(4)}\mathcal{R})/æ$$

— purely geometrical Einstein tensor, which satisfies $\mathrm{div}\, G \equiv 0$ because of Bianchi identities! Then from (16) automatically follows $\mathrm{div}\, \mathcal{T}^{(t)} \equiv 0$ and we obtain well known statement: "motion equations are contained in the field equations of GR". So, we can conclude, that within our approach, fundamental principles of classical mechanics can be associated with the *special* ($\sigma = 1/2$) *elastic properties of space-time.*

Some other considerations on physical nature of local hyperbolicity of space-time and role of boundary conditions in GR, relation between g and Ξ variational principles, some cosmological applications and 4D elastic formulation of classical solids dynamics have been discussed in [5, 6, 7] as well .

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